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Topos Theory and Consistent Histories: The Internal Logic of the Set of All Consistent Sets

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A major problem in the consistent-histories approach to quantum theory is contending with the potentially large number of consistent sets of history propositions. One possibility is to find a scheme in which a unique set is selected in some way. However, in this paper the alternative approach is considered in which all consistent sets are kept, leading to a type of 'many-world-views' picture of the quantum theory. It is shown that a natural way of handling this situation is to employ the theory of varying sets (presheafs) on the space $\mathfrak{B}$ of all nontrivial Boolean subalgebras of the orthoalgebra $\mathcal{UP}$ of history propositions. This approach automatically includes the feature whereby probabilistic predictions are meaningful only in the context of a consistent set of history propositions. More strikingly, it leads to a picture in which the 'truth values' or 'semantic values' of such contextual predictions are not just two-valued (i.e., true and false) but instead lie in a larger logical algebra—a Heyting algebra—whose structure is determined by the space $\mathfrak{B}$ of Boolean subalgebras of $\mathcal{UP}$. This topos-theoretic structure thereby gives a coherent mathematical framework in which to understand the internal logic of the many-world-views picture that arises naturally in the approach to quantum theory based on the ideas of consistent histories.

1. INTRODUCTION

The consistent-histories approach to standard quantum theory was pioneered by Griffiths (1984), Omnès (1988a–c, 1989, 1990, 1992), Gell-Mann and Hartle (1990a,b), and Hartle (1991, 1995) and was motivated in part by a desire to find an interpretation of quantum theory that is less instrumentalist than is that of the standard 'Copenhagen' view. Such a move is particularly desirable in the context of quantum cosmology, where any reference to an 'external observer' seems singularly inappropriate. At a more technical level, the consistent-histories scheme provides an attractive framework in which

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to develop any views on quantum gravity where the microstructure of space-time itself is deemed to be the subject of quantum effects. In particular, questions about the probabilities of various 'generalized' space-times being realized in the universe are difficult to pose and analyze in the more traditional approaches to quantum theory.

The central idea in the consistent-histories scheme is that (i) under certain conditions it is possible to assign probabilities to *history* propositions of a system rather than—as in standard quantum theory—only to propositions concerning properties at a fixed time; and (ii) these probabilities refer to 'the way things are' in some—as yet rather problematic—sense rather than to the results of possible measurements made by an observer from outside the system. A key ingredient in the theory is the 'decoherence function'—a complex-valued function $d(\alpha, \beta)$ of history propositions α, β that measures the 'quantum interference' between them. A complete[2] set of history propositions $C := \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is said to be (strongly) *d-consistent* if $d(\alpha_i, \alpha_j) = 0$ for all $i \neq j$, and under these circumstances the probability that α_i will be realized is identified with the real number $d(\alpha_i, \alpha_i)$ (the formalism is such that these numbers always sum to 1 in a consistent set).

Whether or not this scheme really answers the interpretational questions in quantum theory has been much debated; see, for example, Halliwell (1995), Dowker and Kent (1995, 1996), and Kent (1996). The central problem is the existence of many *d*-consistent sets that are mutually incompatible in the sense that they cannot be combined to give a single larger set. An analogous situation arises in standard quantum theory, but there the problem is resolved by the ubiquitous external observer deciding to measure one observable rather than another. However, this option is not available in the history framework and the problem must be addressed in some way that is internal to the theory itself.

A priori there are two quite different ways of approaching this plethora of *d*-consistent sets. The first is to try to select a unique set that is 'realized' in the actual physical world. Such an approach is typical of those versions of the 'many worlds'—or, in some views, 'many minds'—interpretation[3] of standard quantum theory in which a preferred basis in the Hilbert space of states is used to select one special branch.

The second option is to accept the plethora of *d*-consistent sets as a new type of many-worlds or, perhaps better, 'many-world-views' interpretation of quantum theory. We shall show how the mathematical structure of the collection of all complete sets of history propositions can be exploited to

[2] A set of history propositions is *complete* if (i) the physical realization of any one of them necessarily excludes all the others; and (ii) one of them must be realized.
[3] For a recent review see Butterfield (1995).

provide a novel logic with which to interpret the probabilistic predictions of the theory in the many-world-views context where all d-consistent sets are treated on a par. This involves a type of logical algebra in which probabilistic propositions are (i) manifestly 'contextualized' to a complete set (which need not necessarily be d-consistent) and (ii) not simply binary-valued (i.e., not just true or false). However, this algebra is *not* a 'quantum logic' in the usual sense of the phrase since it *is* distributive. On the other hand, it is not a simple classical Boolean algebra either; rather, it is an example of an 'intuitionistic' logic.

Of course, the idea that probabilistic assertions must be made in the context of a *d-consistent* set is not new—the theme has run through the entire development of the consistent-histories program and, in particular, has been reemphasized recently by Griffiths (1993, 1996). However, allowing the context to be a general complete set and, more dramatically, the use of a 'multivalued' logic are new departures—although, as I hope to show, they follow naturally from the mathematical structure of the consistent-histories formalism. Indeed, the basic idea is easy enough to state, although its ramifications lead at once to concepts drawn from the sophisticated branch of mathematics known as 'topos[4] theory.'

To see how topos ideas arise, suppose that C is a complete set of history propositions that is *not* d-consistent: what would be the status of a probabilistic prediction made in this context? One response is "none at all," but a more physically appropriate observation is that even if C is not itself d-consistent, it might admit a coarse-graining C' (i.e., a set of propositions, each of which is a sum of propositions in C) that *is* d-consistent and in which, therefore, the probabilistic prediction is meaningful; in other words, by agreeing to use less precise propositions, we may arrive at a situation where probabilities *can* be assigned meaningfully. However, we then note that (i) any further coarse-grainings of C' will also be d-consistent and (ii) there may be many such initial choices C' and the same holds for further coarse-grainings of any of them. In the language of topos theory this is expressed by saying that the collection of all d-consistent coarse-grainings of C forms a *sieve* on C with respect to the partial ordering induced by coarse-graining. The main idea is to assign this sieve as the 'truth value'—or, perhaps better, the 'meaning' or 'semantic value'—of a proposition in the context of C. The crucial fact that underpins this suggestion is that the set of all such sieves does indeed form a logical algebra, albeit one that contains more than just the values 'true' and 'false.'

The natural occurrence of sieves in the consistent-histories scheme is the primary motivation for claiming that topos theory—especially the theory

[4] A topos is a special type of category. The relevant details are given further on in this paper.

of sets varying over a partially ordered set—is the natural mathematical tool with which to probe the internal logic of this particular approach to quantum theory. Fortunately, for our purposes it is not necessary to delve too deeply[5] into this—rather abstract—branch of mathematics, and to facilitate the exposition the paper starts with a short summary of some of the relevant ideas about varying sets. This is followed by a discussion of the crucial poset[6] of all nontrivial Boolean subalgebras of the quantum algebra of history propositions. Then comes the heart of the paper, where the appropriate sets of semantic values for propositions in the consistent-histories program are investigated. Some less central technical material is relegated to the appendices.

2. THE TOPOS OF VARYING SETS

2.1. Second-Level Propositions

We begin with the simplest of remarks. In standard set theory, to each subset A of a set X there is associated a 'characteristic map' $\chi^A: X \to \{0, 1\}$ defined by

$$\chi^A(x) := \begin{cases} 1 & \text{if } \; x \in A \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

so that

$$A = (\chi^A)^{-1}\{1\} \tag{2.2}$$

Conversely, any function $\chi: X \to \{0, 1\}$ defines a unique subset $A_\chi := \chi^{-1}\{1\}$ of X whose characteristic function is equal to χ.

Next, consider a hypothetical classical theory whose basic ingredient is a Boolean lattice $\mathcal{UP}$ of propositions about the physical universe.[7] A 'pure state' σ of the system will give rise to a *valuation* on $\mathcal{UP}$, i.e., a homomorphism $V^\sigma: \mathcal{UP} \to \Omega$ from $\mathcal{UP}$ to the simplest Boolean algebra $\Omega := \{0, 1\}$ with '0' interpreted as 'false' and '1' as 'true.' Thus a valuation is a characteristic map that is also a homomorphism between Boolean algebras.

Now let us consider what a probabilistic version of such a theory might look like. In theories with a realist flavor—as is, arguably, the case with the consistent-histories program—there is a temptation (to which I shall succumb) to interpret probability in the sense of 'propensity' rather than in terms

[5] I have deliberately avoided any serious use of category language in the main text of the paper, but have added some more technical remarks in the footnotes.

[6] The word 'poset' is an abbreviation for 'partially ordered set.'

[7] By 'universe' I mean the physical world with all its spatiotemporal aspects. Thus we are talking about 'generalized histories,' not states of affairs at a fixed time.

of (intersubjective) states of knowledge or relative frequency in repeated measurements. In particular, the proposition "$\alpha \in \mathcal{UP}$ is true with probability p" (to be denoted by $\langle\alpha, p\rangle$) is to be read as saying that the state of affairs represented by α has an 'intrinsic tendency' to occur that is measured by the number $p \in [0, 1]$. Thus $\langle\alpha, p\rangle$ is to be construed as being about the universe 'itself' in some way rather than, in particular, our knowledge of the universe or the results of sequences of measurements. A proposition of this type will be labeled 'second-level' by which I mean simply that it is a proposition about the universe that itself involves a proposition $\alpha \in \mathcal{UP}$.

At a mathematical level, we observe that to each probability measure μ on $\mathcal{UP}$ (a 'statistical state' of the system) and for each $p \in [0, 1]$ there is associated the subset of all $\alpha \in \mathcal{UP}$ such that $\mu(\alpha) = p$. In turn, this gives rise to the characteristic map $\chi^{\mu,p}: \mathcal{UP} \to \{0, 1\}$ defined by

$$\chi^{\mu,p}(\alpha) := \begin{cases} 1 & \text{if } \mu(\alpha) = p \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

as a particular example of the situation represented by (2.1).

Note that the characteristic map in (2.3) is *not* a valuation on $\mathcal{UP}$—no role is played by the Boolean structure on $\mathcal{UP}$, which, in this situation, is regarded purely as a *set*. On the other hand, we can think of the second-level propositions $\langle\alpha, p\rangle$ as generating a new logical algebra with respect to which each measure μ on $\mathcal{UP}$ produces a genuine $\{0, 1\}$-valued valuation V^{μ} defined by

$$V^{\mu}(\langle\alpha, p\rangle) := \begin{cases} 1 & \text{if } \mu(\alpha) = p \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

Thus, for example, the conjunction operation on these new propositions is *defined*[8] to be such that, for all μ,

$$V^{\mu}(\langle\alpha, p\rangle \wedge \langle\beta, q\rangle) := \begin{cases} 1 & \text{if } \mu(\alpha) = p \text{ and } \mu(\beta) = q \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

This leads naturally to the idea of two second-level propositions being μ-*equivalent* if their V^{μ} valuations are equal, and *semantically equivalent* if

[8] More formally, the second-level propositions $\langle\alpha, p\rangle$ can be viewed as the sentence letters of a formal language whose sentences are defined recursively by the operations of conjunction, disjunction, and logical implication. The $\{0, 1\}$-valued function V^{μ} on the set Φ_0 of sentence letters is then extended inductively to the set Φ of sentences through successive applications of rules of the type exemplified by (2.5); see Goldblatt (1984) for further discussion of this way of looking at things.

they are μ-equivalent for all measures μ. For example, for all μ and all $p \in [0, 1]$ we have

$$V^{\mu}(\langle\alpha, p\rangle) = V^{\mu}(\langle\neg\alpha, 1 - p\rangle) \tag{2.6}$$

since $\mu(\alpha) + \mu(\neg\alpha) = 1$ for all $\alpha \in \mathcal{UP}$. Hence $\langle\alpha, p\rangle$ and $\langle\neg\alpha, 1 - p\rangle$ are semantically equivalent for all $p \in [0, 1]$. A more complex example is given by the result that, for any disjoint propositions α and β (i.e., $\alpha \wedge \beta = 0$),

$$V^{\mu}(\langle\alpha \vee \beta, p\rangle) = V^{\mu}\left(\bigvee_{q\in[0,1]} \langle\alpha, p - q\rangle \wedge \langle\beta, q\rangle\right) \tag{2.7}$$

which arises from the fact that $\mu(\alpha \vee \beta) = \mu(\alpha) + \mu(\beta)$ for any such pair of propositions. Thus we see that, if $\alpha \wedge \beta = 0$, then $\langle\alpha \vee \beta, p\rangle$ and $\bigvee_{q\in[0,1]} \langle\alpha, p - q\rangle \wedge \langle\beta, q\rangle$ are semantically equivalent for all $p \in [0, 1]$.

The situation in the consistent-histories program is similar in many respects. Once again, there is an algebra $\mathcal{UP}$ of 'universe propositions,' although—as part of a quantum theory—it is no longer Boolean. There are also second-level propositions of the type $\langle\alpha, p\rangle$, although the role of a probability measure μ: $\mathcal{UP} \to \mathbb{R}$ is now taken by a decoherence function d: $\mathcal{UP} \times \mathcal{UP} \to \mathbb{C}$. However, the really significant new features of the consistent-histories theory are that (i) a proposition $\langle\alpha, p\rangle$ is physically meaningful only in the *context* of a d-consistent set (or, as we shall see, any complete set) of histories, and (ii) as we shall show, the associated truth values, or semantic values, can be regarded as lying in an algebra that is larger than $\{0, 1\}$.

2.2. Sets Through Time

As an example[9] of how contexts and generalized semantic values can arise, consider a fixed set X of people who are all alive at some initial time and whose bodies are preserved once they die (and who are still referred to as 'people' in that state). Thus if $D(t) \subseteq X$ denotes the subset of dead people at any time t, then as t increases, $D(t)$ will clearly stay constant or increase, i.e., $t_1 \leq t_2$ implies $D(t_1) \subseteq D(t_2)$. Such a parametrized family of sets $D(t)$, $t \in \mathbb{R}$, is an example of what has been called a "set through time" by those working in the foundations of topos theory (for example, Lawvere, 1975; Goldblatt, 1984; Bell, 1988; MacLane and Moerdijk, 1992).

Now suppose that some members of our population are—in fact—immortal. Suppose also that the members of X are all philosophers with a nostalgic leaning toward logical positivism. Then what truth value should be

[9] A similar example has been explored by Dummett (1959) in the form of the proposition, "A city will never be built on this spot"; I thank Jeremy Butterfield for this observation.

assigned to the proposition "person x is mortal" if all truth statements are required to be verifiable in some operational sense? If death has already occurred by the time the proposition is asserted, then, of course, the proposition is true (assuming that the deadness of a body is something that can be confirmed operationally). However, if x is alive, the proposition cannot be said to be true—on the assumption that mortality of a living being cannot be verified operationally—but neither can it be denied, since even if x *is* numbered among the immortals, there is no way of showing this. Thus we are led to the notion of a 'stage of truth' as a context in which a proposition acquires meaning—in our case, the time t—and to the idea that the truth values of a statement at a stage t may not just lie in the set $\{0, 1\}$.

Of course, a dedicated verificationist might simply insist that the proposition "x is mortal" is meaningless if asserted at a time t_0 when x is not dead. However, topos theory provides a more positive answer that stems from the observation that there may be a later time t at which x *does* die, and then of course $x \in D(t')$ for all times $t' \geq t$. A key idea in the theory of sets-through-time is that the 'truth value'—or, perhaps better, the 'meaning' or '*semantic value*'—at the stage t_0 of the proposition "x is mortal" is *defined* to be the set $\chi^D_{t_0}(x)$ of all later times[10] t at which x is dead:

$$\chi^D_{t_0}(x) := \{t \geq t_0 \,|\, x \in D(t)\} \tag{2.8}$$

Note that if x never dies, i.e., if he or she is immortal, then the right-hand side of (2.8) is just the empty set. On the other hand, x is dead at a time t if and only if

$$\chi^D_t(x) = \uparrow(t) := [t, \infty) \tag{2.9}$$

Equivalently, at stage t,

$$D(t) = (\chi^D_t)^{-1}\{\uparrow(t)\} \tag{2.10}$$

When compared with (2.2), the relation (2.10) shows that the parametrized family of maps $\chi^D_{t_0}: X \to \Omega(t_0)$, $t_0 \in \mathbb{R}$—where $\Omega(t_0)$ denotes the collection of all upper sets lying above t_0 (i.e., sets of the form $[s, \infty)$, $s \geq t_0$)—is the analogue of the single characteristic function of normal set theory.

From a logical perspective, the crucial property of this set $\Omega(t_0)$ of all possible semantic values at stage t_0 is that it possesses the structure of a Heyting algebra. Thus (i) $\Omega(t_0)$ is a distributive lattice under the usual set-theoretic operations of union and intersection, and (ii) $\Omega(t_0)$ has the property

[10] The example gets a little artificial at this point in the sense that a transtemporal view of the history of the population X is needed for the semantic values to be appreciated: clearly a job for one of the immortals!

that for any $a, b \in \Omega(t_0)$ there is a unique element $(a \Rightarrow b) \in \Omega(t_0)$ (with the intuitive meaning "if a, *then* b") satisfying, for all $c \in \Omega(t_0)$,

$$c \leq (a \Rightarrow b) \quad \text{if and only if} \quad c \wedge a \leq b \tag{2.11}$$

The negation operation in such an algebra is defined by $\neg a := (a \Rightarrow 0)$, and satisfies the relation[11] $a \leq \neg\neg a$ for all a.[12] Indeed, it can be shown that a Heyting algebra is Boolean if and only if $a = \neg\neg a$ for all a.

2.3. Sets Varying over a Partially Ordered Set

A key fact for our program is that the ideas sketched above extend readily to the situation where the 'stages of truth' are elements of a general partially ordered set $\mathcal{P}$ (for example, Goldblatt, 1984; Bell, 1988). In our case, $\mathcal{P}$ will be the set of nontrivial Boolean subalgebras of the space of quantum history propositions with $W_1 \leq W_2$ defined to mean $W_2 \subseteq W_1$ (so that W_2 is a 'coarse-graining' of W_1). The necessary mathematical development is most naturally expressed in the language of category theory, although for our purposes all that is really needed is the idea that a category consists of a collection of things called 'objects'—mathematical entities with some precisely defined internal structure—and 'morphisms' between pairs of such objects, where a morphism is a type of structure-preserving 'map' (but not necessarily in the sense of set theory).

The relevant category for us is the category $\mathrm{Set}^{\mathcal{P}}$ of "varying sets over $\mathcal{P}$." Here, an object X is defined to be an assignment to each $p \in \mathcal{P}$ of a set $X(p)$, and an assignment to each pair $p \leq q$ of a map $X_{pq}: X(p) \to X(q)$ such that (i) X_{pp} is the identity map on $X(p)$, and (ii) the relation

$$X_{qr} \circ X_{pq} = X_{pr} \tag{2.12}$$

is satisfied whenever $p \leq q \leq r$.[13]

A *morphism* $\eta: Y \to X$ between two such objects Y, X in $\mathrm{Set}^{\mathcal{P}}$ is defined to be a family of maps $\eta_p: Y(p) \to X(p)$, $p \in \mathcal{P}$, satisfying the compatibility conditions[14]

$$\eta_q \circ Y_{pq} = X_{pq} \circ \eta_p \tag{2.13}$$

[11] A classic example of a Heyting algebra is the collection of all open sets for a topological space X. In particular, if A is an open set, then the negation $\neg A$ is defined as the *interior*, $\mathrm{int}(X - A)$, of $X - A$. It is clear that $A \subseteq \mathrm{int}(X - \mathrm{int}(X - A))$.

[12] This is one of the main reasons why a Heyting algebra is chosen as the formal mathematical structure that underlies intuitionistic logic. Thus there is a strong connection between the theory of sets through time and the logic of intuitionism.

[13] In the context of category theory, what is being exploited here is the familiar fact that a poset $\mathcal{P}$ can be regarded as a category in its own right in which (i) the objects are the elements of $\mathcal{P}$ and (ii) there are no morphisms $p \to q$ unless $p \leq q$, in which case there is just one. A 'varying set over $\mathcal{P}$' is then just a functor from the category $\mathcal{P}$ to the category 'Set' of normal sets. This is closely related to the concept of a 'presheaf' on $\mathcal{P}$.

[14] If $\mathcal{P}$ is regarded as a category, then η is a 'natural transformation' between the functors Y and X.

as shown in the commutative diagram

$$\begin{array}{ccc} Y(p) & \xrightarrow{Y_{pq}} & Y(q) \\ \Big\downarrow \eta_p & & \Big\downarrow \eta_q \\ X(p) & \xrightarrow{X_{pq}} & X(q) \end{array} \tag{2.14}$$

In particular, a *subobject* of a varying set[15] $X = \{X(p), p \in \mathcal{P}\}$ is a varying set $A = \{A(p), p \in \mathcal{P}\}$ with the property that $A(p) \subseteq X(p)$ for all $p \in \mathcal{P}$, and such that A_{pq}: $A(p) \to A(q)$ is just the restriction of X_{pq}: $X(p) \to X(q)$ to the subset $A(p) \subseteq X(p)$. These relations are captured nicely by the commutative diagram

$$\begin{array}{ccc} A(p) & \xrightarrow{A_{pq}} & A(q) \\ \Big\downarrow & & \Big\downarrow \\ X(p) & \xrightarrow{X_{pq}} & X(q) \end{array} \tag{2.15}$$

where the vertical arrows are subset inclusions.

A simple, but important special case is when the varying set $X(p)$ is in fact constant i.e., $X(p) = X$ for all $p \in \mathcal{P}$, and X_{pq} is the identity map from $X = X(p)$ to $X = X(q)$ for all pairs $p \leq q$. In this situation, each set $A(p)$, $p \in \mathcal{P}$, can be regarded as a subset of the fixed set X, and the condition in (2.15) for a varying set $A := \{A(p), p \in \mathcal{P}\}$ to be a bona fide subobject of X reduces to

$$p \leq q \quad \text{implies} \quad A(p) \subseteq A(q) \tag{2.16}$$

This special case where $X(p)$ is constant also gives rise to the varying-set analogue of a 'complement' of a subset. The obvious family of subsets of X to serve as the complement of $\{A(p), p \in \mathcal{P}\}$ is $\{X - A(p), p \in \mathcal{P}\}$, but this does not give a proper varying set, since $p \leq q$ implies $X - A(p) \supseteq X - A(q)$, which is the wrong behavior. It turns out that the appropriate definition is $\neg A := \{\neg A(p), p \in \mathcal{P}\}$, where

$$\neg A(p) := \{x \in X \mid \forall q \geq p, x \notin A(q)\} \tag{2.17}$$

which is immediately checked to be a genuine varying set, i.e., $p \leq q$ implies

[15] The notation does not include a specific reference to the functions X_{pq}: $X(p) \to X(q)$, but these are understood to be implicitly included.

$\neg A(p) \subseteq \neg A(q)$. It follows that $x \notin \neg A(p)$ if and only if there is some $q \geq p$ such that $x \in A(q)$, and hence

$$\neg\neg A(p) := \{x \in X \mid \forall q \geq p\; \exists r \geq q \text{ s.t. } x \in A(r)\} \tag{2.18}$$

It is clear that $A(p) \subseteq \neg\neg A(p)$, whereas in normal set theory the double complement of a subset is always equal to the subset itself. This nonstandard behavior in the varying-set theory is a reflection of the fact that the underlying logical structure is non-Boolean (see later).

As in the case of sets through time, a key role is played by the collections $\Omega(p)$, $p \in \mathcal{P}$, of all upper sets lying above p. More precisely, a *sieve*[16] on p in $\mathcal{P}$ is defined to be any subset S of $\mathcal{P}$ such that if $r \in S$, then (i) $r \geq p$ and (ii) $r' \in S$ for all $r' \geq r$. For each $p \in \mathcal{P}$, the set $\Omega(p)$ of all sieves on p can be shown to be a Heyting algebra (under the usual set-theoretic operations of inclusion, intersection, and union of subsets[17]), and for all pairs $p \leq q$ there is a natural map $\Omega_{pq}: \Omega(p) \to \Omega(q)$ defined by

$$\Omega_{pq}(S) := S \cap \uparrow(q) \tag{2.19}$$

where $\uparrow(q) := \{r \in \mathcal{P} \mid r \geq q\}$ is the unit element in the Heyting algebra $\Omega(q)$ (the null element is the empty set). It is easy to see that, with the maps Ω_{pq} in (2.19), $\Omega := \{\Omega(p), p \in \mathcal{P}\}$ is a varying set over $\mathcal{P}$ and hence an object in the category $\mathrm{Set}^{\mathcal{P}}$.

A very important example of the use of Ω occurs if A is a subobject of the object X. There is then an associated *characteristic* morphism $\chi^A: X \to \Omega$ with, at each stage $p \in \mathcal{P}$, the 'component' $\chi^A_p: X(p) \to \Omega(p)$ being defined by

$$\chi^A_p(x) := \{q \geq p \mid X_{pq}(x) \in A(q)\} \tag{2.20}$$

where the fact that the right-hand side of (2.20) actually *is* a sieve on p in $\mathcal{P}$ follows from the defining properties of a subobject. Thus in each 'branch' of the poset going up from p, $\chi^A_p(x)$ picks out the first member q (the "time till truth") in that branch for which $X_{pq}(x)$ lies in the subset $A(q)$, and the commutative diagram (2.15) then guarantees that $X_{pr}(x)$ will lie in $A(r)$ for all $r \geq q$. In the special case where $X(p) = X$ for all p, (2.20) simplifies to [cf. (2.8)]

$$\chi^A_p(x) := \{q \geq p \mid x \in A(q)\} \tag{2.21}$$

In what follows, the expression (2.21) plays a crucial role as the analogue in the theory of varying sets of the characteristic map (2.1) $\chi^A: X \to \{0, 1\}$

[16]This is the notation employed by Bell (1988); other authors (for example, MacLane and Moerdijk, 1992) use the term 'cosieve' for what Bell calls a 'sieve' and vice versa.

[17]The precise algebraic relations will be given later for the specific example of interest in the consistent-histories theory.

of normal set theory. Indeed, the analogue of the relation (2.2) for the situation epitomized by (2.21) is [cf. (2.10)]

$$A(p) = (\chi_p^A)^{-1}\{(\uparrow(p)\} \tag{2.22}$$

at each stage $p \in \mathcal{P}$. Conversely, each morphism χ: $X \to \Omega$ defines a subobject of X [via (2.22)], and for this reason the object Ω in $\mathrm{Set}^{\mathcal{P}}$ is known as the *subobject classifier* in the category $\mathrm{Set}^{\mathcal{P}}$; the existence of such an object is one of the defining properties[18] for a category to be a topos, which $\mathrm{Set}^{\mathcal{P}}$ is. As the target of characteristic maps (i.e., the analogue of $\{0, 1\}$ in normal set theory), Ω can be thought of as the 'object of truth values'—a soubriquet that is reinforced by the observation that Ω has the internal structure of a Heyting algebra. For example, the conjunction $\wedge$: $\Omega \times \Omega \to \Omega$ is defined to be the morphism in the category $\mathrm{Set}^{\mathcal{P}}$ whose components $\wedge_p$: $\Omega(p) \times \Omega(p) \to \Omega(p)$, $p \in \mathcal{P}$, are the conjunction operations (the set-theoretic intersection of sieves on p) in the 'local' Heyting algebras $\Omega(p)$; the other logical operations are defined in a similar way.

The main thesis of the present paper is that a situation closely analogous to the one sketched above arises naturally in the theory of consistent histories where the basic quantum ingredients are an orthoalgebra $\mathcal{UP}$ of history propositions and a specific decoherence function d. In particular, we have (i) the idea of a 'context' or a 'stage' and (ii) the property that—at each such stage—the truth values lie in a Heyting algebra. As emphasized already, the key point in the formalism of consistent histories is that the central second-level propositions $\langle\alpha, p\rangle$ ("the probability of the history proposition $\alpha \in \mathcal{UP}$ is p") only have a physical meaning in the context of a particular d-consistent set of propositions to which α belongs or, to be more precise (see later), in the context of any set of propositions that can be coarse-grained to give a d-consistent set that contains α. It is technically convenient to employ the Boolean subalgebra of $\mathcal{UP}$ generated by any given set of history propositions, rather than the set itself, and in this framework my thesis is essentially that:

- each nontrivial Boolean subalgebra W_0 of the set $\mathcal{UP}$ of all history propositions can serve as a possible 'stage'
- the truth value (or semantic value) of $\langle\alpha, p\rangle$ at a particular stage W_0 is related to the collection of all coarse-grainings W of W_0 that contain α and are d-consistent.

As we shall see, the implementation of this idea involves a specific application of the idea of sets varying over a poset, and hence we do indeed

[18] Another defining property for a category $\mathcal{C}$ to be a topos is that a product $A \times B$ exists for any pair of objects A, B in $\mathcal{C}$. For the full definition see one of the standard texts (for example, Goldblatt, 1984; Bell, 1988; MacLane and Moerdijk, 1992).

obtain a Heyting algebra of possible semantic values at each stage. Moreover, we will show how second-level propositions of the type $\langle\alpha, p\rangle$ can be associated with Ω-valued morphisms; as such, they belong to the internal logic (and, indeed, formal language) that is associated with the topos $\mathrm{Set}^{\mathcal{B}}$, where $\mathcal{B}$ denotes the poset of all nontrivial Boolean subalgebras of $\mathcal{UP}$ (see below for details). Thus the internal logic of the topos provides a framework for understanding the logical structure of probabilistic predictions in a consistent-histories theory in a way that automatically includes all possible contextual references to *d*-consistent sets. We thereby arrive at a coherent logical structure for this particular 'many-world-views' picture of quantum theory.

3. BOOLEAN SUBALGEBRAS OF PROPOSITIONS

3.1. The General Formalism of Consistent Histories

In the general approach to the consistent-history formalism developed by Isham (1994) and Isham and Linden (1994), the central mathematical ingredient is a pair ($\mathcal{UP}$, $\mathfrak{D}$) where $\mathcal{UP}$ is an orthoalgebra[19] of 'history propositions' and $\mathfrak{D}$ is the space of decoherence functions defined on this algebra [for a short summary of the scheme see Isham (1995)].

It should be emphasized from the outset that, in practice, the orthoalgebra formalism is much less abstract than it might appear at first. In particular, for any given physical system it is always appropriate to consider the possibility that $\mathcal{UP}$ may simply be the algebra $P(\mathcal{V})$ of projection operators on some Hilbert space $\mathcal{V}$; in this case, $\alpha \oplus \beta$ is defined if and only if $\alpha\beta = 0$, and then $\alpha \oplus \beta = \alpha + \beta$. For example, it was shown in Isham (1994) that the history version of normal quantum theory for, say, a finite number of time points $\{t_1, t_2, \ldots, t_n\}$ can be cast into this form. Specifically, the history propositions are identified as projection operators on the tensor product space $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n}$, where each $\mathcal{H}_{t_i}$ is a copy of the Hilbert space of states $\mathcal{H}$ of standard canonical quantum theory. The extension of this idea to a continuous time variable is discussed in Isham and Linden (1995).

Returning to the general formalism, we recall that a *decoherence function* is a map d: $\mathcal{UP} \times \mathcal{UP} \to \mathbb{C}$ that satisfies the following conditions:

[19] An orthoalgebra $\mathcal{UP}$ is a partially ordered set with greatest element 1 and least element 0 and for which there is a notion of when two elements α, β are *orthogonal*, denoted $\alpha \perp \beta$. If α and β are such that $\alpha \perp \beta$, then they can be combined to give a new element $\alpha \oplus \beta \in \mathcal{UP}$. Furthermore, $\alpha \leq \beta$ if and only if $\beta = \alpha \oplus \gamma$ for some $\gamma \in \mathcal{UP}$. There is also a negation operation with $\alpha \oplus \neg\alpha = 1$ [for the full definition of an orthoalgebra see Foulis *et al.* (1992)]. It should be noted that the structure of an orthoalgebra is much weaker than that of a lattice. In the latter there are two connectives $\wedge$ and $\vee$, both of which are defined on *all* pairs of elements, unlike the single, partial operation $\oplus$ in an orthoalgebra. A lattice is a special type of orthoalgebra with $\alpha \oplus \beta$ being defined on disjoint lattice elements α, β (i.e., those for which $\alpha \leq \neg\beta$) as $\alpha \oplus \beta := \alpha \vee \beta$.

1. *Hermiticity:* $d(\alpha, \beta) = d(\beta, \alpha)^*$ for all α, β.
2. *Positivity:* $d(\alpha, \alpha) \geq 0$ for all α.
3. *Additivity:* if $\alpha \perp \beta$ (i.e., α and β are orthogonal), then, for all γ, $d(\alpha \oplus \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma)$.
4. *Normalization:* $d(1, 1) = 1$.

Note that the additivity condition implies that, for all $\alpha \in \mathcal{UP}$,

$$d(0, \alpha) = 0 \tag{3.1}$$

We also note that, as shown by Isham *et al.* (1994), in the concrete case where $\mathcal{UP} = P(\mathcal{V})$ for some Hilbert space $\mathcal{V}$ every decoherence function can be written in the form

$$d(\alpha, \beta) = \mathrm{tr}_{\mathcal{V} \otimes \mathcal{V}}(\alpha \otimes \beta X) \tag{3.2}$$

where X belongs to a precisely specified class of operators on $\mathcal{V} \otimes \mathcal{V}$.

Following Gell-Mann and Hartle, a finite set $C := \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ of nonzero propositions is said to be *complete* if (i) $\alpha_i \perp \alpha_j$ for all $i, j = 1, 2, \ldots, N$; (ii) the elements of C are 'jointly compatible,' i.e., they belong to some Boolean subalgebra of $\mathcal{UP}$; and (iii) $\alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_N = 1$. In algebraic terms, a complete set of history propositions is simply a finite *partition of unity* in the orthoalgebra $\mathcal{UP}$.

It should be noted that in the history version of standard quantum theory, the decoherence function for a particular system depends on both the initial state and the Hamiltonian. Thus, in general, for any specific history system the decoherence function d will be one particular element of $\mathfrak{D}$. It must be emphasized that only *d-consistent* sets of history propositions are given an immediate physical interpretation. A complete set C of history propositions is said to be[20] *d-consistent* if $d(\alpha, \beta) = 0$ for all $\alpha, \beta \in C$ such that $\alpha \neq \beta$. Under these circumstances $d(\alpha, \alpha)$ is regarded as the *probability* that the history proposition α is true. The axioms above then guarantee that the usual Kolmogoroff probability rules are satisfied on the Boolean algebra generated by C.

It is worth noting that the idea of an orthoalgebra is closely related to that of a *Boolean manifold*: an algebra that is 'covered' by a collection of maximal Boolean subalgebras with appropriate compatibility conditions on any pair that overlap (Hardegree and Frazer, 1982). Being Boolean, these subalgebras of propositions carry a logical structure that is essentially classical, a feature of the consistent-histories scheme that was focal in the seminal work of Griffiths and Omnès and that has been reemphasized recently by

[20] In what follows I shall only consider the strong case where $d(\alpha, \beta)$ itself vanishes, rather than just the real part of $d(\alpha, \beta)$. The phrase 'consistent set' will always mean 'strongly' consistent in this sense.

Griffiths (1993, 1996). In the approach outlined above, these Boolean algebras are glued together from the outset to form an orthoalgebra $\mathcal{UP}$ of propositions from which the physically interpretable subsets are selected by the consistency conditions with respect to a chosen decoherence function.

Any partition of unity $C := \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ generates a Boolean algebra whose elements are the finite[21] $\oplus$-sums of elements of C (hence the elements of C are atoms of this algebra). If C is d-consistent, and if $\alpha := \oplus_{i\in I_1}\alpha_i$ and $\beta := \oplus_{j\in I_2}\alpha_j$ are two members of the algebra (where I_1 and I_2 are subsets of the index set $\{1, 2, \ldots, N\}$), then, by the additivity property of the decoherence function d,

$$d(\alpha, \beta) = \sum_{i\in I_1\cap I_2} d(\alpha_i, \alpha_i) \tag{3.3}$$

On the other hand

$$\alpha \wedge \beta = \bigoplus_{i\in I_1\cap I_2} \alpha_i \tag{3.4}$$

and so

$$d(\alpha, \beta) = d(\alpha \wedge \beta, \alpha \wedge \beta) \tag{3.5}$$

for all α, β in C.

In a general orthoalgebra $\mathcal{UP}$ not every Boolean subalgebra is necessarily generated by a partition of unity—for example, any Boolean subalgebra that is not atomic falls in this class. Partly for this reason it is helpful to define consistency for Boolean algebras per se, rather than go via partitions of unity. Moreover, it is also pedagogically useful to do so, as it emphasizes the essentially 'classical' nature of the properties of the propositions in a d-consistent set. Motivated by (3.5), I propose the following definition:

Definition 3.1. For a given history system $(\mathcal{UP}, d)$, where $d \in \mathcal{D}$, a Boolean subalgebra W of $\mathcal{UP}$ is *d-consistent* if for all $\alpha, \beta \in W$ we have $d(\alpha, \beta) = d(\alpha \wedge \beta, \alpha \wedge \beta)$.

Note that the smallest subalgebra $\{0, 1\}$ is trivially d-consistent for any d since, by virtue of (3.1), $d(0, 1) = 0 = d(0, 0) = d(0 \wedge 1, 0 \wedge 1)$. It is technically convenient to exclude this trivial Boolean subalgebra as a possible consistent set (it contains no interesting propositions) and this will be done in what follows. The set of all nontrivial Boolean subalgebras of $\mathcal{UP}$ will be denoted $\mathcal{B}$; the set of all nontrivial d-consistent Boolean subalgebras will be denoted $\mathcal{B}^d$.

[21] With appropriate care these ideas can be generalized to include countable sets and sums, but the details are not important here.

The relation of the definition above to the earlier one of strong consistency is contained in the following lemma.

Lemma. The condition (3.5) on all elements α, β in a Boolean subalgebra W is equivalent to

$$d(\alpha, \beta) = 0 \qquad \text{for all} \quad \alpha, \beta \in W \quad \text{such that} \quad \alpha \perp \beta \tag{3.6}$$

Proof. Suppose that (3.5) is true and let $\alpha \perp \beta$. Then $\alpha \wedge \beta = 0$ (in a Boolean algebra, $\alpha \perp \beta$ if and only if $\alpha \wedge \beta = 0$) and hence $d(\alpha, \beta) = d(0, 0) = 0$.

Conversely, suppose (3.6) is true and let $\alpha, \beta \in W$. Then there exist jointly orthogonal elements $\alpha_1, \beta_1, \gamma \in W$ such that $\alpha = \alpha_1 \oplus \gamma$ and $\beta = \beta_1 \oplus \gamma$. Indeed, we can choose $\alpha_1 := \alpha \wedge \neg\beta$, $\beta_1 := \beta \wedge \neg\alpha$, and $\gamma := \alpha \wedge \beta$, where the general lattice operation $\wedge$ is well defined in this Boolean subalgebra. Then, by the additivity property of the decoherence function,

$$\begin{aligned} d(\alpha, \beta) &= d(\alpha_1 \oplus \gamma, \beta_1 \oplus \gamma) = d(\alpha_1, \beta_1) + d(\alpha_1, \gamma) + d(\gamma, \beta_1) + d(\gamma, \gamma) \\ &= d(\alpha \wedge \beta, \alpha \wedge \beta) \end{aligned} \tag{3.7}$$

since (3.6) means that $\alpha_1 \perp \beta_1$ implies $d(\alpha_1, \beta_1) = 0$, etc. QED

In what follows I shall refer to a Boolean subalgebra of $\mathcal{UP}$ as a *window* in order to convey the idea that it affords a potential way of 'looking' at the physical world; the initial letter 'w' also serves to remind us that a window can be regarded as a possible 'world-view' or even 'weltanschauung.' A d-consistent window is what Griffiths (1996) calls a 'framework.'

3.2. Key Features of the Space $\mathcal{B}$ of Boolean Subalgebras

The crucial property of $\mathcal{B}$ for our purposes is that it is a partially ordered set with respect to subset inclusion, and we write $W_1 \leq W_2$ if $W_1 \supseteq W_2$ ($W_1 \leq W_2$ is defined in this way rather than as $W_1 \subseteq W_2$ in order to be consistent with our earlier discussion of sets varying over a poset). For such a pair we say that (i) W_1 is a *fine-graining* of W_2 and (ii) W_2 is a *coarse-graining* of W_1 (for convenience, this terminology includes the idea that any window $W \in \mathcal{B}$ is both a coarse-graining and a fine-graining of itself). Note that the set $\mathcal{B}^d$ of d-consistent windows is a subset of $\mathcal{B}$ and inherits its partial-ordering structure. Moreover, $\mathcal{B}^d$ is closed under coarse-graining: if W_1 is d-consistent, then so is any $W_2 \geq W_1$; thus coarse-graining preserves d-consistency—a crucial property for our later discussions. However, there is generally no biggest (with respect to $\subseteq$) d-consistent coarse-graining of a non-d-consistent window.

Let us now recall the earlier discussion of sets varying over a poset while making the following key definitions.

Definition 3.2. 1. A *sieve* on $W_0 \in \mathcal{B}$ in $\mathcal{B}$ is a (possibly empty) subset S of windows in $\mathcal{B}$ such that

(a) $W \in S$ implies $W \geq W_0$ (i.e., $W \subseteq W_0$)

(b) $W \in S$ and $W' \geq W$ (i.e., $W' \subseteq W$) implies $W' \in S$.

Thus a sieve on W_0 is an *upper set* in $(\mathcal{B}, \leq)$ all of whose elements are coarse-grainings of W_0.

The set of all sieves on W_0 is denoted $\Omega(W_0)$.

2. A sieve S on W_0 is *d-consistent* if every $W \in S$ is d-consistent.

The following properties of sieves are crucial for our purposes.

1. For each W_0, the set $\Omega(W_0)$ of all sieves on W_0 in $\mathcal{B}$ is a partially ordered set with $S_1 \leq S_2$ being defined as $S_1 \subseteq S_2$. The greatest element 1 is the *principal* sieve

$$\uparrow(W_0) := \{W \in \mathcal{B} \mid W \geq W_0\} \equiv \{W \in \mathcal{B} \mid W \subseteq W_0\} \tag{3.8}$$

and the least element 0 is the *empty* subset of windows.

2. The poset $\Omega(W_0)$ is a distributive lattice with the operations (i) $S_1 \wedge S_2 := S_1 \cap S_2$ and (ii) $S_1 \vee S_2 := S_1 \cup S_2$. In fact, $\Omega(W_0)$ is a *Heyting algebra*, i.e., given sieves S_1 and S_2, there is a sieve $(S_1 \Rightarrow S_2)$ such that, for all S,

$$S \leq (S_1 \Rightarrow S_2) \qquad \text{if and only if} \quad S \wedge S_1 \leq S_2 \tag{3.9}$$

This sieve is defined as

$$(S_1 \Rightarrow S_2) := \{W \subseteq W_0 \mid \forall W' \subseteq W \text{ if } W' \in S_1 \text{ then } W' \in S_2\} \tag{3.10}$$

In a Heyting algebra, the negation of an element x is defined by $\neg x := (x \Rightarrow 0)$. Thus, for a sieve S on W_0,

$$\neg S := \{W \subseteq W_0 \mid \forall W' \subseteq W, W' \notin S\} \tag{3.11}$$

As explained earlier, a central idea in the internal logic of varying sets is that the Heyting algebra $\Omega(W_0)$ serves as the space of *semantic values* for propositions at the *stage* W_0.

3. The collection $\Omega := \{\Omega(W), W \in \mathcal{B}\}$ is a set varying over $\mathcal{B}$ under the definition, for all pairs $W_1 \leq W_2$ (i.e., $W_2 \subseteq W_1$),

$$\Omega_{W_1W_2}\colon \quad \Omega(W_1) \rightarrow \Omega(W_2)$$
$$S \mapsto S \cap \uparrow(W_2) := \{W \subseteq W_2 \mid W \in S\} \tag{3.12}$$

We can also define the sets of d-consistent sieves

$$\Omega^d(W) := \{S \in \Omega(W) \mid S \text{ is } d\text{-consistent}\} \tag{3.13}$$

and note that, like Ω, $\Omega^d := \{\Omega^d(W), W \in \mathcal{B}\}$ is a set varying over $(\mathcal{B}, \leq)$

if, for $W_1 \leq W_2$, $\Omega^d_{W_1W_2}: \Omega^d(W_1) \to \Omega^d(W_2)$ is defined by $\Omega^d_{W_1W_2}(S) := S \cap \uparrow(W_2)$.

Finally, although no explicit use of it will be made here, we note that another simple example of a set varying over $(\mathcal{B}, \leq)$ is given by $\{\mathcal{D}_W, W \in \mathcal{B}\}$, where $\mathcal{D}_W$ is defined to be the set of all decoherence functions for which W is a d-consistent window:

$$\mathcal{D}_W := \{d \in \mathcal{D} \mid W \in \mathcal{B}^d\} \tag{3.14}$$

The family $\{\mathcal{D}_W, W \in \mathcal{B}\}$ is a genuine object in $\mathrm{Set}^{\mathcal{B}}$ since if W_2 is a coarse-graining of W_1 (i.e., $W_1 \leq W_2$), then if d is such that W_1 is d-consistent (i.e., $d \in \mathcal{D}_{W_1}$), then W_2 is d-consistent, too (i.e., $d \in \mathcal{D}_{W_2}$), and hence $W_1 \leq W_2$ implies that $\mathcal{D}_{W_1} \subseteq \mathcal{D}_{W_2}$.

4. SEMANTIC VALUES IN CONSISTENT HISTORIES

4.1. Realizable Propositions

We come now to the main task of the paper: to formulate precisely the idea that a second-level proposition like $\langle\alpha, p\rangle$ ("the probability of the history proposition α being true is p") has a meaning only in the context of a window, and has a semantic value that belongs to some logical structure associated with that window—in particular, we anticipate that a semantic value can be identified with a sieve on the window.

Let us start by considering what is necessary for a proposition $\langle\alpha, p\rangle$ to have any meaning at all in the context of a particular window W and for a given decoherence function d. Perhaps the simplest position to adopt here is that in order to be able to 'realize' α in the context of W, the history proposition α must belong to the Boolean algebra W, and W must be d-consistent.[22] This suggests the following definition:

Definition 4.1. A proposition $\alpha \in \mathcal{UP}$ is *d-realizable in a window W* if (i) W is d-consistent and (ii) $\alpha \in W$.

Then

$$R^d(W) := \{\alpha \mid W \in \mathcal{B}^d \text{ and } \alpha \in W\} = \begin{cases} W & \text{if } W \in \mathcal{B}^d \\ \emptyset & \text{otherwise} \end{cases} \tag{4.1}$$

is the set of all[23] propositions that are d-realizable in W.

[22] Of course, it could be that $d(\alpha, \alpha) = 0$, but the statement that the state of affairs described by the history proposition α has *zero* probability is still a positive prediction about the universe.

[23] This definition has the property that even the 0 and 1 history propositions are deemed not to be d-realizable in a window that is not d-consistent. Of course, this does affect the fact that, for all $d \in \mathcal{D}$, $d(0, 0) = 0$, $d(0, 1) = 0$, and $d(1, 1) = 1$.

Continuing to reason in this heuristic way, we could argue next that even if a window W_0 is *not* d-consistent, the proposition $\langle\alpha, p\rangle$ still has a meaning at stage W_0 provided that a coarse-graining W of W_0 exists in which α *is* d-realizable. However, there may be many such coarse-grainings: and the focal idea of this paper is that this should be reflected by assigning an appropriate *semantic value* to $\langle\alpha, p\rangle$—in the present case, a natural choice would be the set of all coarse grainings of W_0 in which α *is* d-realizable [on the assumption that $d(\alpha, \alpha) = p$; if not, the semantic value is deemed to be the empty set]. In other words, we tentatively assign to the second-level proposition $\langle\alpha, p\rangle$ the semantic value at stage W_0 defined as

$$V^d_{W_0}(\langle\alpha, p\rangle) := \begin{cases} \{W \subseteq W_0 \mid W \in \mathcal{B}^d \text{ and } \alpha \in W\} & \text{if } d(\alpha, \alpha) = p \\ \emptyset & \text{otherwise} \end{cases} \tag{4.2}$$

which would make sense provided that the set of all such semantic values belongs to some logical algebra.

However, this assignment does not work in the way we have been anticipating because the right-hand side of (4.2) is generally *not* a sieve on W_0 [because if W belongs to $V^d_{W_0}(\langle\alpha, p\rangle)$, then any $W' \subset W$ with $\alpha \notin W'$ will *not*]. Thus we cannot identify the set of possible semantic values at a stage W_0 with the Heyting algebra of the space of sieves on W_0. In itself this does not rule out the use of (4.2) but it implies that any logical structure on the set of semantic values must be obtained in a way that is different from our anticipated use of the topos of varying sets $\mathrm{Set}^{\mathcal{B}}$. One possibility is sketched in Appendix A.

4.2. Accessible Propositions

The problem with the semantic value $V^d_{W_0}\langle\alpha, p\rangle$ suggested in (4.2) can be seen from a somewhat different perspective by noting that $R^d := \{R^d(W), W \in \mathcal{B}\}$ does not define a proper varying set over $\mathcal{B}$. This is because increasing the size of the window W (*i.e.*, fine-graining it) increases the number of propositions contained in W—which suggests that $W_2 \subseteq W_1$ implies $R^d(W_2) \subseteq R^d(W_1)$—but it decreases the chances of d-consistency—which suggests that $W_2 \subseteq W_1$ implies $R^d(W_3) \supseteq R^d(W_1)$. The net effect is that if $W_1 \leq W_2$ there is no obvious relation between $R^d(W_1)$ and $R^d(W_2)$ and hence no obvious candidate for the collection of maps $R^d_{W_1 W_2}$: $R^d(W_1) \to R^d(W_2)$ that is necessary for $R^d = \{R^d(W), W \in \mathcal{B}\}$ to be a varying set.

Note that a genuine varying set *can* be obtained by the simple expedient of replacing the condition $\alpha \in W$ in (4.1) by $\alpha \notin W$. This gives rise to a new concept: namely of a proposition α being 'unrealizable,' but this is not what we are seeking and therefore further discussion is relegated to Appendix

B where it serves as a particularly simple example of what is meant by an object in the topos $\mathrm{Set}^{\mathcal{B}}$.

Evidently a new idea is needed of what it means to say that a proposition α is 'realizable' in a window W. On reflection, do we really require that α actually belongs to W? Surely it suffices that W can be extended (*i.e.*, fine-grained) to a bigger d-consistent window that *does* contain α?

This new concept differs subtly from the earlier version of 'realizability' and the terminology should reflect this. We are thus lead to introduce a new family of second-level propositions of the form "α is d-accessible" where $\alpha \in \mathcal{UP}$. As with the second-level propositions $\langle\alpha, p\rangle$, there is no reference to windows *per se*, and I shall indicate this by referring to propositions of this type as *global*. However, the key idea we wish to develop is that—as hinted above in the example of d-realizability—in order to interpret a global proposition within the framework of the consistent-histories program it is first necessary to 'localize' it by constructing secondary versions that *do* refer to windows (*cf.* (4.2)). Then we can introduce the notion of the 'semantic value at a stage W_0' of the global proposition, and we find that it does indeed lie in a Heyting algebra. The aim is to use the topos-theoretic ideas discussed earlier so that, in particular, the Heyting algebra appropriate to a stage W_0 is the set $\Omega(W_0)$ of sieves on W_0 in $(\mathcal{B}, \leq)$.

We begin with the following definition of the 'localized' version of the new family of global propositions "α is d-accessible," denoted $\langle\alpha, A^d\rangle$.

Definition 4.2. A proposition $\alpha \in \mathcal{UP}$ is *d-accessible from a window* W if there exists $W' \supseteq W$ such that (i) $\alpha \in W'$; and (ii) W' is d-consistent (*i.e.*, α is d-realizable in W').[24] Then

$$A^d(W) := \{\alpha \mid \exists W' \supseteq W \text{ s.t. } W' \in \mathcal{B}^d \text{ and } \alpha \in W'\} \tag{4.3}$$

is the set of all propositions that are d-accessible from W. Note that this can be rewritten as

$$A^d(W) = \{\alpha \mid \exists W' \supseteq W \text{ s.t. } \alpha \in R^d(W')\} \tag{4.4}$$

where $R^d(W)$ was defined in (4.1). Thus a proposition α is d-accessible from a window W if and only if there exists some *fine-graining* W' of W in which α is d-realizable.

The following properties of these sets are crucial for our purposes.

1. If $W_1 \leq W_2$ then $A^d(W_1) \subseteq A^d(W_2)$ (because coarse-graining preserves the property of d-consistency), and hence, unlike the case for R^d, the collection $A^d := \{A^d(W), W \in \mathcal{B}\}$ is a genuine varying set over $(\mathcal{B}, \leq)$ with $A^d_{W_1W_2}$: $A^d(W_1) \to A^d(W_2)$ defined simply as subset inclusion. In this sense, 'accessibil-

[24] Note that this implies that W itself must be d-consistent—no propositions are accessible from a non-d-consistent window.

ity' works while 'realizability' fails. As we shall see, this new object A^d in $\mathrm{Set}^{\mathcal{B}}$ is the crucial ingredient in my topos-based semantics for interpreting the second-level propositions $\langle\alpha, p\rangle$ in the consistent-histories program.

2. The varying set A^d is a subobject of the constant varying set $\Delta\mathcal{UP}$ in Set^B,

$$\Delta\mathcal{UP}(W) := \mathcal{UP} \text{ for all } W \in \mathcal{B} \tag{4.5}$$

Hence there is a characteristic morphism $\chi^{A^d}: \Delta\mathcal{UP} \to \Omega$ in the topos $\mathrm{Set}^{\mathcal{B}}$ which, according to (2.21), is defined at any stage W_0 by (*cf.*, (2.8))

$$\begin{aligned} \chi^{A^d}_{W_0}: \Delta\mathcal{UP}(W_0) &\to \Omega(W_0) \\ \alpha &\mapsto \{W \geq W_0 \mid \alpha \in A^d(W)\} \\ &= \{W \subseteq W_0 \mid \exists W' \supseteq W \text{ s.t. } W' \in \mathcal{B}^d \text{ and } \alpha \in W'\} \end{aligned} \tag{4.6}$$

where, as can readily be checked, the right hand side is a *bona fide* sieve on W_0.

The sieve on the right hand side of (4.6)—which actually belongs to the subject $\Omega^d(W_0)$ of $\Omega(W_0)$—is interpreted as the semantic value at the stage W_0 of the global proposition "α is d-accessible." Note that, by the property (2.22) of a characteristic morphism, a history proposition α is d-accessible from a window W if and only if

$$\chi^{A^d}_W(\alpha) = \uparrow(W) \tag{4.7}$$

where $\uparrow(W) = \{W' \in \mathcal{B} \mid W \leq W'\}$ is the unit 1 of the Heyting algebra $\Omega(W)$.

The definition (4.6) of $\chi^{A^d}: \Delta\mathcal{UP} \to \Omega$ and property (4.7) illustrate the essentially 'fuzzy' nature[25] of subobjects in $\mathrm{Set}^{\mathcal{B}}$. More precisely, if α is d-accessible from W then $\chi^{A^d}_W(\alpha) = 1$; but even if α is not d-accessible from a window W_0, the proposition $\langle\alpha, A^d\rangle$ is still ascribed a semantic value at stage W_0 that is generically not the null element 0 (the empty sieve) of the Heyting algebra $\Omega(W_0)$: namely, the set of all coarse-grainings W of W_0 from which α *is* d-accessible. Thus the semantic value at stage W_0 of the global proposition "α is d-accessible" reflects the extent to which W_0 needs to be changed in order that α *does* become d-accessible from it. Hence coarse-graining a window is the analogue in the consistent-histories theory of choosing a later time in the example of sets-through-time discussed earlier (compare (4.7) with (2.9)).

A key role for Ω is to impart a logical structure to the collection of all global propositions, and the first step in this direction is to note that (4.6) can be used to define for each $\alpha \in \mathcal{UP}$ (and for given decoherence function d) what is known in the topos literature as a *global element* of the object

[25]This is not a coincidence: fuzzy set theory can be viewed as a sub-branch of topos theory.

Ω, *i.e.*, a morphism $1 \to \Omega$ in $\text{Set}^{\mathcal{B}}$ where 1 is the terminal object[26] in $\text{Set}^{\mathcal{B}}$ defined by $1(W) := \{*\}$ (the set with one element) for all $W \in \mathcal{B}$. Specifically, for each $\alpha \in \mathcal{UP}$ we define $\tilde{\chi}^{A^d}(\alpha): 1 \to \Omega$ by specifying its components $\tilde{\chi}^{A^d}(\alpha)_{W_0}: 1(W_0) \to \Omega(W_0)$ to be

$$\tilde{\chi}^{A^d}(\alpha)_{W_0}(*) := \chi^{A^d}_{W_0}(\alpha) \tag{4.8}$$

where the right hand side is given by (4.6). In turn, this produces a map[27]

$$\tilde{\chi}^{A^d}: \mathcal{UP} \to \text{Him}_{\text{Set}^{\mathcal{B}}}(1, \Omega) \tag{4.9}$$

where $\text{Hom}_{\text{Set}^{\mathcal{B}}}(1, \Omega)$ denotes the set of morphisms from 1 to Ω in the topos category $\text{Set}^{\mathcal{B}}$.

By these means, to each global proposition $\langle\alpha, A^d\rangle$ we can associate a corresponding 'valuation' morphism

$$V\langle\alpha, A^d\rangle: 1 \to \Omega \tag{4.10}$$

where $V\langle\alpha, A^d\rangle := \tilde{\chi}^{A^d}(\alpha)$ is a global element of Ω; *i.e.*, V is a map from global propositions to global elements. In normal set theory, a map from $\{*\}$ (the terminal object in the category of sets) to a set X picks out a unique element of X, and (4.10) can be regarded as the analog of this procedure in the category $\text{Set}^{\mathcal{B}}$ of varying sets. Thus (4.10) encapsulates the idea that in our topos interpretation of the consistent-histories formalism, a 'generalized truth value' is associated to each global proposition $\langle\alpha, A^d\rangle$—namely the global element $V\langle\alpha, A^d\rangle: 1 \to \Omega$ of Ω.

Referring to (4.10) as a 'valuation' seems to imply that it preserves some logical structure on the propositions $\langle\alpha, A^d\rangle$. However, we do not have any such structure *a priori*: rather, the intention of (4.10) is to use the Heyting algebra structure of Ω to *define* a logical algebra on the global propositions $\langle\alpha, A^d\rangle$—a goal that can be achieved by associating each such second-level proposition with the corresponding global element of Ω. For example, if $V\langle\alpha, A^d\rangle: 1 \to \Omega$ and $V\langle\beta, A^d\rangle: 1 \to \Omega$ are global elements of Ω corresponding to the global propositions $\langle\alpha, A^d\rangle$ and $\langle\beta, A^d\rangle$ respectively then the global

[26] An object 1 is said to be a *terminal object* in a category if there is just one morphism from any other object to 1; it is easy to see that any two terminal objects are isomorphic. In the category of sets a terminal object is any set $\{*\}$ with just a single element. In this case a morphism is just a map, and hence a morphism $\{*\} \to X$ picks out a unique element of X.

[27] We are exploiting here the fact that the constant presheaf functor Δ: Set $\to \text{Set}^{\mathcal{B}}$ is left adjoint to the 'global sections' functor Γ: $\text{Set}^{\mathcal{B}} \to$ Set where, for any object F in $\text{Set}^{\mathcal{B}}$, we have $\Gamma F := \text{Hom}_{\text{Set}^{\mathcal{B}}}(1, F)$. This adjointness relation gives rise to a natural isomorphism $\text{Hom}_{\text{Set}^{\mathcal{B}}}(\Delta S, F) \simeq \text{Hom}_{\text{Set}}(S, \Gamma F)$ for any set S. In our case the set S is $\mathcal{UP}$ and the functor F is Ω; thus the isomorphism of interest is $\text{Hom}_{\text{Set}^{\mathcal{B}}}(\Delta\mathcal{UP}, \Omega) \simeq \text{Hom}_{\text{Set}}(\mathcal{UP}, \Gamma\Omega)$. The element in $\text{Hom}_{\text{Set}^{\mathcal{B}}}(\Delta\mathcal{UP}, \Omega)$ with which we are concerned is χ^{A^d} and its image in $\text{Hom}_{\text{Set}}(\mathcal{UP}, \Gamma\Omega)$ is what we have denoted $\tilde{\chi}^{A^d}$. Thus $\tilde{\chi}^{A^d}(\alpha) \in \Gamma\Omega = \text{Hom}_{\text{Set}^{\mathcal{B}}}(1, \Omega)$.

proposition "$\langle\alpha, A^d\rangle$ *and* $\langle\beta, A^d\rangle$" is associated[28] with the global element of Ω defined by the chain

$$1 \overset{<V\langle\alpha,A^d\rangle,V\langle\beta,A^d\rangle>}{\longrightarrow} \Omega \times \Omega \overset{\wedge}{\rightarrow} \Omega \tag{4.11}$$

where $\wedge : \Omega \times \Omega \rightarrow \Omega$ is the 'and' operation in the Heyting algebra structure on Ω. This is a rather sophisticated analog of the treatment of $\langle\alpha, p\rangle \wedge \langle\beta, q\rangle$ by the expression (2.5) in the context of our earlier discussion of second-level propositions in a classical theory.

Note however that the map $\tilde{\chi}^{A^d}$ in (4.9) is not one-to-one, and hence neither is the valuation map $\langle\alpha, A^d\rangle \mapsto V\langle\alpha, A^d\rangle$ that associates a global element of Ω with each $\langle\alpha, A^d\rangle$. Thus—analogous again to our discussion in section 2.1 of classical second-level propositions—we are led to define two global proposition as being *d-semantically equivalent* if they are associated with the same global element of Ω (with a given decoherence function d): properly speaking, it is only to the equivalence classes of such propositions that the logical algebra applies.

For example, although in the construction (4.6) the quantity $\Delta\mathcal{UP}$ is regarded purely as a *set* and the orthoalgebra structure plays no *a priori* role, nevertheless—since W is a Boolean subalgebra of $\mathcal{UP}$—if $\alpha \in W$ then $\neg\alpha \in W$ and *vice versa*. Hence $\alpha \notin W$ if and only if $\neg\alpha \notin W$, which implies that, for all $\alpha \in \mathcal{UP}$,

$$\chi^{A^d}_{W_0}(\alpha) = \chi^{A^d}_{W_0}(\neg\alpha) \tag{4.12}$$

Thus $\langle\alpha, A^d\rangle$ and $\langle\neg\alpha, A^d\rangle$ are d-semantically equivalent[29] for any $\alpha \in \mathcal{UP}$ and for all decoherence functions d.

4.3. The Semantic Values for $\langle\alpha, p\rangle$

Finally we are in a position to treat the main goal of the paper, namely propositions of the type $\langle\alpha, p\rangle$—"the probability of history proposition α being true is p." All we have to do is to supplement the requirement of d-accessibility with the additional condition $d(\alpha, \alpha) = p$. Thus I propose to interpret the global proposition $\langle\alpha, p\rangle$ by specifying it to have the following 'localized' form:

Definition 4.3. The proposition $\langle\alpha, p\rangle$ is *d-accessible from a window W* if (i) α is d-accessible from W; and (ii) $d(\alpha, \alpha) = p$.

[28] Note that "$\langle\alpha, A^d\rangle$ *and* $\langle\beta, A^d\rangle$" is not itself of the form $\langle\gamma, A^d\rangle$ for any $\gamma \in \mathcal{UP}$. It is thus more accurate to think of the propositions $\langle\alpha, A^d\rangle$, $\alpha \in \mathcal{UP}$ as the sentence letters of a formal language whose sentences are constructed with the aid of the usual logical connectives.

[29] I am grateful to Pen Maddy for the gnomic remark that this conclusion is consistent with proposition 4.0621 in Wittgenstein's *Tractatus*.

Let $A^{d,p}(W)$ denote the set of all propositions $\alpha \in \mathcal{UP}$ such that $\langle\alpha, p\rangle$ is d-accessible from W:

$$A^{d,p}(W) := \{\alpha \mid \exists W' \supseteq W \text{ s.t. } W' \in \mathcal{B}^d, \alpha \in W', \text{and } d(\alpha, \alpha) = p\} \quad (4.13)$$

These sets obey the basic condition $W_1 \leq W_2$ implies $A^{d,p}(W_1) \subseteq A^{d,p}(W_2)$ and hence, for each $p \in [0, 1]$, $A^{d,p} := \{A^{d,p}(W), W \in \mathcal{B}\}$ is a varying set over $\mathcal{B}$.

The varying set $A^{d,p}$ is a subobject of the constant varying set $\Delta\mathcal{UP}$, and hence for each decoherence function d (an analog of the state σ that arises in (2.3)) and each $p \in [0, 1]$ we get the crucial characteristic morphism $\chi^{d,p}: \Delta\mathcal{UP} \to \Omega$ in $\mathrm{Set}^{\mathcal{B}}$ whose components are the maps $\chi^{d,p}_{W_0}: \Delta\mathcal{UP}(W_0) \to \Omega(W_0)$, $W_0 \in \mathcal{B}$, defined by

$$\chi^{d,p}_{W_0}(\alpha) := \begin{cases} \{W \subseteq W_0 \mid \exists W' \supseteq W \text{ with } W' \in \mathcal{B}^d \text{ and } \alpha \in W'\} & \text{if } d(\alpha, \alpha) = p \\ \emptyset & \text{otherwise} \end{cases} \quad (4.14)$$

The right hand side of (4.14) is a genuine sieve and is to be regarded as the semantic value at stage W_0 of the global proposition $\langle\alpha, p\rangle$ "the state of affairs represented by the history proposition $\alpha \in \mathcal{UP}$ has probability p of occurring."

As was the case with $\langle\alpha, A^d\rangle$, the global proposition $\langle\alpha, p\rangle$ can be associated with a global element $\tilde{\chi}^{d,p}(\alpha): 1 \to \Omega$ whose components $\tilde{\chi}^{d,p}(\alpha)_{W_0}: 1(W_0) \to \Omega(W_0)$ are defined as $\tilde{\chi}^{d,p}(\alpha)W_0(*) :\rangle \chi^{d,p}_{W_0}(\alpha)$. Putting together these various results we finally arrive at the desired 'valuation morphism'

$$V^d\langle\alpha, p\rangle: 1 \to \Omega \quad (4.15)$$

whose components $V^d\langle\alpha, p\rangle_{W_0}: 1(W_0) \to \Omega(W_0)$ are given by

$$V^d\langle\alpha, p\rangle_{W_0}(*) = \begin{cases} \{W \subseteq W_0 \mid \exists W' \supseteq W \text{ s.t. } W' \in \mathcal{B}^d \text{ and } \alpha \in W'\} & \text{if } d(\alpha, \alpha) = p \\ \emptyset & \text{otherwise} \end{cases} \quad (4.16)$$

The topos result (4.16) should be compared with the simple expression (2.4) that applies in a more conventional probability theory. Once again we see the 'fuzzy-set' nature of the topos scheme in the sense that at any particular stage W_0 the proposition $\langle \alpha, p \rangle$ may be assigned a semantic value other than $0 := \emptyset$ or $1 := \uparrow(W_0)$.

As in the earlier example of the second-level propositions $\langle \alpha, A^d \rangle$, it is appropriate to define two global propositions $\langle \alpha, p \rangle$ and $\langle \beta, q \rangle$ to be '*d*-semantically-equivalent' if they are associated with the same global element of Ω, *i.e.*, if their semantic values are equal in all windows. For example, it is clear from (4.16) that, for all decoherence functions d, the second-level propositions $\langle \alpha, p \rangle$ and $\langle \neg\alpha, 1 - p \rangle$ are d-semantically equivalent for all $\alpha \in \mathcal{UP}$ and all $p \in [0, 1]$. This is because if α belongs to some window W' then so does $\neg\alpha$. Furthermore, if W' is d-consistent then $d(\alpha, \neg\alpha) = 0$ and hence, by additivity of the decoherence function d,

$$1 = d(1, 1) = d(\alpha \oplus \neg\alpha, \alpha \oplus \neg\alpha) = d(\alpha, \alpha) + d(\neg\alpha, \neg\alpha) \quad (4.17)$$

which shows that $d(\neg\alpha, \neg\alpha) = 1 - d(\alpha, \alpha)$.

By these means, the (equivalence classes of) elements of the formal language generated by global propositions of the type $\langle \alpha, p \rangle$ can be associated with elements of a logical algebra that is identified as a subalgebra of the Heyting algebra on the set $\mathrm{Hom}_{\mathrm{Set}^{\mathcal{B}}}(1, \Omega)$ of global elements of Ω in $\mathrm{Set}^{\mathcal{B}}$. The expression (4.15)—with its component version (4.16)—represents the final form of our analysis of the logical structure of the consistent-history propositions "the probability that $\alpha \in \mathcal{UP}$ is realized is p" in the context of topos theory.

5. CONCLUSIONS

A key ingredient in the consistent-histories formulation of quantum theory is the existence of d-consistent sets of propositions. We have argued that, in the approach where a preferred set is *not* selected once and for all, the ensuing many-world-views semantics can be described mathematically with a topos-theoretic framework based on the idea of sets varying over the partially ordered set $\mathcal{B}$ of all nontrivial Boolean subalgebras of the orthoalgebra of history propositions. In particular, we have seen how a global proposition such as "the probability of the history proposition α being true is p" can be interpreted in a way that identifies any window $W_0 \in \mathcal{B}$ as a potential 'stage' and where the semantic values at each such stage lie in a Heyting algebra. This situation stems from the central claim that each global proposition can be identified with a global element $1 \rightarrow \Omega$ of the space Ω of semantic values in the topos category $\mathrm{Set}^{\mathcal{B}}$. Propositions of this type in the consistent-histories program include $\langle \alpha, A^d \rangle$, $\langle \alpha, p \rangle$, and (as discussed in Appendix B), $\langle \alpha, U^d \rangle$. The collection of d-semantic equivalence classes of all[30] such

[30]Mixed propositions—for example, "$\langle \alpha, U^d \rangle$ *and* $\langle \beta, p \rangle$"—are allowed.

propositions generates a logical structure that is inherited from that of Ω. It should be emphasized once more that, in practice, the space $\mathcal{UP}$ may simply be the set of projection operators on some Hilbert space, in which case the analysis of the crucial poset $\mathcal{B}$ is a viable concrete task.

The general conclusion of this paper is that topos methods provide a natural mathematical framework in which to discuss the inner logical structure that lies behind ideas of many windows, or world views, in the quantum theory of histories. One general aim of this approach is to avoid the instrumentalism that dominates much conventional thinking about quantum theory, although—as is frequently the case—it is difficult to give a simple physical picture of what the formalism means in these circumstances. However, if we accept the idea that 'classical realism' is associated in some way with a Boolean algebra of propositions, then we have to say that—because of its intrinsic Heyting algebra logic—the 'many-windows' interpretation of the decoherent-histories formalism corresponds to a type of neorealism that, on the one hand, is more complicated and subtle than the simple realism of classical physics, but which, on the other hand, does not go as far as the nondistributive structure that characterizes quantum logic proper. Of course, this does not affect the fact that the underlying orthoalgebra $\mathcal{UP}$ of history propositions *is* a genuine quantum logic.

In the context of 'many world-views' it is worth noting that the concept of a proposition being 'd-accessible' from a window W clearly extends to Boolean subalgebras in general: i.e., we can say that a window W' is d-accessible from a window W if there exists a d-consistent window W'' that contains both W' and W; a single proposition α is then d-accessible from W in the sense of (4.3) if and only if the window $\{0, 1, \alpha, \neg\alpha\}$ is d-accessible from W in the sense just described. However, this is just the consistent-histories analogue of the idea of 'relative possibility' introduced by Kripke (1963) in his original study of the semantics of intuitionistic logic (see also Loux, 1979). This suggests that modal concepts such as 'necessity' or 'possibility' should find a natural home in the quantum formalism of consistent histories after making the shift from Kripkean 'worlds' to 'windows'; but this remains a topic for future work.

The topos-theoretic ideas used in the present paper are rather elementary, being essentially restricted to the theory of presheafs on a poset, and there is a lot more to the subject than that. However, even at the simple level of the theory of varying sets it seems clear that the ideas discussed here could find applications in other areas of quantum theory where some type of contextuality arises. An example might be the idea of 'relational quantum theory' that has been actively developed recently by several authors (for example, Crane, 1995; Smolin, 1995; Rovelli, 1997). It also seems possible that the well-known contextuality of truth values in standard quantum theory (i.e., the Kochen–Specker theorem) could be explored profitably from this perspective.

APPENDIX A. AN ALTERNATIVE APPROACH

I shall sketch here a method whereby it *is* possible to assign the set [cf. (4.2)]

$$\Gamma^d_\alpha(W_0) := \{W \subseteq W_0 \mid W \in \mathcal{B}^d \text{ and } \alpha \in W\} \tag{A.1}$$

as the semantic value at stage W_0 of the global proposition $\langle\alpha, p\rangle$ [on the supposition that $d(\alpha, \alpha) = p$] even though the right-hand side of (A.1) is not a sieve on W_0 in $(\mathcal{B}, \leq)$.

The first observation is that—assuming for simplicity that $d(\alpha, \alpha) = p$ and $d(\beta, \beta) = q$—if $\Gamma^d_\alpha(W_0)$ and $\Gamma^d_\beta(W_0)$ *were* to be the semantic values of propositions $\langle\alpha, p\rangle$ and $\langle\beta, q\rangle$, respectively, at stage W_0 then the global proposition "$\langle\alpha, p\rangle$ *or* $\langle\beta, q\rangle$" would presumably be represented at stage W_0 by the set

$$\begin{aligned} &\Gamma^d_\alpha(W_0) \cup \Gamma^d_\beta(W_0) \\ &\quad = \{W \subseteq W_0 \mid W \in \mathcal{B}^d, \alpha \in W\} \cup \{W \subseteq W_0 \mid W \in \mathcal{B}^d, \beta \in W\} \\ &\quad = \{W \subseteq W_0 \mid W \in \mathcal{B}^d \text{ and } (\alpha \in W \text{ or } \beta \in W)\} \\ &\quad = \{W \subseteq W_0 \mid W \in \mathcal{B}^d \text{ and } \{\alpha, \beta\} \cap W \neq \emptyset\} \end{aligned} \tag{A.2}$$

Since the right-hand side of (A.2) is not itself of the form $\Gamma^d_\gamma(W_0)$, we do not have algebraic closure. However, the structure of (A.2) is suggestive and leads to the idea of defining the 'trapped' sets

$$T^d_F(W_0) := \{W \subseteq W_0 \mid W \in \mathcal{B}^d \text{ and } F \cap W \neq \emptyset\} \tag{A.3}$$

where F is any finite set of propositions from $\mathcal{UP}$. Note that $\Gamma^d_\alpha(W_0) \equiv T^d_{\{\alpha\}}(W_0)$.

The collection of all such sets is closed under the union operation since

$$T^d_F(W_0) \cup T^d_G(W_0) = T^d_{F\cup G}(W_0) \tag{A.4}$$

although under intersections we have

$$\begin{aligned} &T^d_F(W_0) \cap T^d_G(W_0) = \\ &\quad \{W \subseteq W_0 \mid W \in \mathcal{B}^d \text{ and } (F \cap W \neq \emptyset \;\&\; G \cap W \neq \emptyset)\} \end{aligned} \tag{A.5}$$

Closure can be reestablished by defining the collections of subalgebras

$$T^d_{F_1, F_2, \ldots, F_m}(W_0) = \{W \subseteq W_0 \mid W \in B_d \text{ and } (F_1 \cap W \neq \emptyset \,\&\, F_2 \cap W \neq \emptyset, \ldots, \,\&\, F_m \cap W \neq \emptyset)\} \tag{A.6}$$

where $F_1, F_2, \ldots, F_m$ is any finite collection of finite sets of propositions in $\mathcal{UP}$.

A simple way of using these sets to generate a logical algebra follows from the following observation. There is a well-known topology (the Vietoris topology) that can be placed on the set of all closed subsets of a topological space and which involves trapping sets, rather as in (A.3). Motivated by what is done in the Vietoris situation, the natural procedure in our case is to define a topology τ_d on $\mathcal{B}$ by taking as a subbasis the collection of all sets of the form $T^d_F(W)$ as F ranges over all finite subsets of $\mathcal{UP}$ and W ranges over all nontrivial Boolean subalgebras of $\mathcal{UP}$. The open sets of the topological space $(\mathcal{B}, \tau_d)$ can then serve as the semantic values of our system.

Notice that this procedure does indeed produce a logical structure since the collection of open sets in any topological space is always a Heyting algebra. However, this is rather far from our original idea of presheafs on the partially ordered set $(\mathcal{B}, \leq)$ and needs to be treated as a separate theory.

APPENDIX B. UNREALIZABLE PROPOSITIONS

The 'localized' form of a new, and rather simple, global proposition "α is d-unrealizable" (denoted $\langle \alpha, U^d \rangle$) is given by the following definition.

Definition 2.1. A proposition $\alpha \in \mathcal{UP}$ is *d-unrealizable in a window* W if (i) W is d-consistent and (ii) $\alpha \notin W$.

Thus we can define

$$U^d(W) := \{\alpha \mid W \in \mathcal{B}^d \text{ and } \alpha \notin W\} = \begin{cases} \mathcal{UP} - W & \text{if } W \in \mathcal{B}^d \\ \emptyset & \text{if } W \notin \mathcal{B}^d \end{cases} \tag{B.1}$$

as the set of all propositions[31] that are d-unrealizable in the window W. We note that:

1. If $W_1 \leq W_2$, then $U^d(W_1) \subseteq U^d(W_2)$ and hence, unlike the case for R^d, the collection $U^d := \{U^d(W), W \in \mathcal{B}\}$ is a genuine varying set over the poset $(\mathcal{B}, \leq)$ if the maps $U^d_{W_1W_2}$: $U^d(W_1) \to U^d(W_2)$, $W_1 \leq W_2$, are defined to be the subset inclusions.

[31] Note that, according to this definition, the 0 and 1 history propositions are never d-unrealizable.

2. The object U^d in $\mathrm{Set}^{\mathcal{B}}$ can be regarded as a subobject of the constant varying set $\Delta\mathcal{UP}$ in $\mathrm{Set}^{\mathcal{B}}$. The associated characteristic morphism is

$$\chi^{U^d}_{W_0}:\quad \Delta\mathcal{UP}(W_0) \to \Omega(W_0)$$

$$\alpha \mapsto \{W \geq W_0 \mid \alpha \in U^d(W)\}$$

$$= \{W \subseteq W_0 \mid W \in \mathcal{B}^d \text{ and } \alpha \notin W\} \qquad \text{(B.2)}$$

As was the case with $\langle\alpha, A^d\rangle$, the new global proposition $\langle\alpha, U^d\rangle$ can be associated with a global element of the Heyting algebra Ω via the morphism

$$\tilde{\chi}^{U^d}(\alpha):\quad 1 \to \Omega \qquad \text{(B.3)}$$

whose components are defined by $\tilde{\chi}^{U^d}(\alpha)_{W_0}(*) := \chi^{U^d}_{W_0}(\alpha)$ [cf. (4.8)]. There is an associated valuation morphism $V\langle\alpha, U^d\rangle$: $1 \to \Omega$, where $V\langle\alpha, U^d\rangle := \tilde{\chi}^{U^d}(\alpha)$. We also note that, analogously to (4.12),

$$\chi^{U^d}_{W_0}(\alpha) = \chi^{U^d}_{W_0}(\neg\alpha) \qquad \text{(B.4)}$$

for all windows W_0 and all $\alpha \in \mathcal{UP}$. Thus $\langle\alpha, U^d\rangle$ and $\langle\neg\alpha, U^d\rangle$ are d-semantically equivalent for any $\alpha \in \mathcal{UP}$ and for all decoherence functions d.

Finally we remark that one might have tried to use these results to resolve the 'realizability' problem by defining a proposition $\alpha \in \mathcal{UP}$ to be d-realizable in a window if it is not d-unrealizable there. This involves taking the negation[32] of the variable set U^d in the appropriate Heyting algebra of subobjects of the constant variable set $\Delta\mathcal{UP}$. The result is the variable set $\neg U^d := \{\neg U^d(W),\ W \in \mathcal{B}\}$, where

$$\neg U^d(W) = \{\alpha \mid \forall W' \geq W, \alpha \notin U^d(W')\}$$

$$= \{\alpha \mid \forall W' \subseteq W, W' \notin \mathcal{B}^d \text{ or } \alpha \in W'\} \qquad \text{(B.5)}$$

However, this does not seem to capture at all what we instinctively want to be true for a 'realizable' proposition and the definition given in the text seems far more appropriate.

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[32] Simply taking the set-theoretic complement of each $U^d(W)$, $W \in \mathcal{B}$, will not produce a proper element of $\mathrm{Set}^{\mathcal{B}}$ since the resulting sets $\{\alpha \mid \alpha \notin U^d(W)\}$ have the wrong behavior with respect to $W_1 \subseteq W_2$. This was mentioned earlier in the context of (2.17).

REFERENCES

Bell, J. L. (1988). *Toposes and Local Set Theories: An Introduction,* Clarendon Press, Oxford.

Butterfield, J. (1995). Worlds, minds and quanta, *Proceedings of the Aristotelian Society,* **69**, 113–158.

Crane, L. (1995). Clocks and categories: Is quantum gravity algebraic? *Journal of Mathematical Physics,* **36**, 6180–6193.

Dowker, H. F., and Kent, A. (1995). Properties of consistent histories, *Physical Review Letters,* **75**, 3038–3041.

Dowker, H. F., and Kent, A. (1996). On the consistent histories approach to quantum mechanics, *Journal of Statistical Physics,* **82**, 1575–1646.

Dummett, M. (1959). Truth, *Proceedings of the Aristotelian Society,* **59**, 141–162.

Foulis, D. J., Greechie, R. J., and Rüttimann, G. T. (1992). Filters and supports in orthoalgebras, *International Journal of Theoretical Physics,* **31**, 789–807.

Gell-Mann, M., and Hartle, J. (1990a). Alternative decohering histories in quantum mechanics, in *Proceedings of the 25th International Conference on High Energy Physics, Singapore, August, 1990,* K. K. Phua and Y. Yamaguchi, eds., World Scientific, Singapore.

Gell-Mann, M., and Hartle, J. (1990b). Quantum mechanics in the light of quantum cosmology, in *Complexity, Entropy and the Physics of Information,* W. Zurek, ed., Addison-Wesley, Reading, Massachusetts.

Goldblatt, R. (1984). *Topoi: The Categorial Analysis of Logic,* North-Holland, Amsterdam.

Griffiths, R. B. (1984). Consistent histories and the interpretation of quantum mechanics, *Journal of Statistical Physics,* **36**, 219–272.

Griffiths, R. B. (1993). *Foundations of Physics,* **23**, 1601.

Griffiths, R. B. (1996). Consistent histories and quantum reasoning [quant-ph/9606004].

Halliwell, J. (1995). A review of the decoherent histories approach to quantum mechanics, in *Fundamental Problems in Quantum Theory,* D. M. Greenberger and A. Zeilinger, eds., New York Academy of Sciences, New York.

Hardegree, G. M., and Frazer, P. J. (1982). Charting the labyrinth of quantum logics: A progress report, in *Current Issues in Quantum Logic,* E. G. Beltrametti and B. V. van Frassen, eds., Plenum Press, New York.

Hartle, J. (1991). The quantum mechanics of cosmology, in *Quantum Cosmology and Baby Universes,* S. Coleman, J. Hartle, T. Piran, and S. Weinberg, eds., World Scientific, Singapore.

Hartle, J. (1995). Spacetime quantum mechanics and the quantum mechanics of space-time, in *Proceedings of the 1992 Les Houches School, Gravitation and Quantization,* B. Julia and J. Zinn-Justin, eds., Elsevier Science, Amsterdam.

Isham, C. J. (1994). Quantum logic and the histories approach to quantum theory, *Journal of Mathematical Physics,* **35**, 2157–2185.

Isham, C. J. (1995). Quantum logic and decohering histories, in *Topics in Quantum Field Theory,* D. H. Tchrakian, ed., World Scientific, Singapore.

Isham, C. J., and Linden, N. (1994). Quantum temporal logic and decoherence functionals in the histories approach to generalized quantum theory, *Journal of Mathematical Physics,* **35**, 5452–5476.

Isham, C. J., and Linden, N. (1995). Continuous histories and the history group in generalized quantum theory, *Journal of Mathematical Physics,* **36**, 5392–5408.

Isham, C. J., Linden, N., and Schreckenberg, S. (1994). The classification of decoherence functionals: An analogue of Gleason's theorem, *Journal of Mathematical Physics,* **35**, 6360–6370.

Kent, A. (1996). Consistent sets contradict [gr-qc/9604012].
Kripke, S. (1963). Semantical considerations on modal logic, *Acta Philosophica Fennica*, **16**, 83–94.
Lawvere, F. W. (1975). Continuously variable sets: Algebraic geometry = geometric logic, in *Proceedings Logic Colloquium Bristol 1973*, North-Holland, Amsterdam.
Loux, M. J. (1979). *The Possible and the Actual*, Cornell University Press, Ithaca, New York.
MacLane, S., and Moerdijk, I. (1992). *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Springer-Verlag, Berlin.
Omnès, R. (1988a). Logical reformulation of quantum mechanics. I. Foundations, *Journal of Statistical Physics*, **53**, 893–932.
Omnès, R. (1988b). Logical reformulation of quantum mechanics. II. Interferences and the Einstein–Podolsky–Rosen experiment, *Journal of Statistical Physics*, **53**, 933–955.
Omnès, R. (1988c). Logical reformulation of quantum mechanics. III. Classical limit and irreversibility, *Journal of Statistical Physics*, **53**, 957–975.
Omnès, R. (1989). Logical reformulation of quantum mechanics. IV. Projectors in semiclassical physics, *Journal of Statistical Physics*, **57**, 357–382.
Omnès, R. (1990). From Hilbert space to common sense: A synthesis of recent progress in the interpretation of quantum mechanics, *Annals of Physics*, **201**, 354–447.
Omnès, R. (1992). Consistent interpretations of quantum mechanics, *Reviews of Modern Physics*, **64**, 339–382.
Rovelli, C. (1996). Relational quantum theory, *International Journal of Theoretical Physics*, in press.
Smolin, L. (1995). The Bekenstein bound, topological quantum field theory, and pluralistic quantum cosmology [gr-qc/9508064].
Wittgenstein, L. (1966). *Tractatus Logico-Philosophicus*, Routledge & Kegan Paul, London.